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Successive Differentiation and Leibnitz Theorem

Lecture - 03

Important formula:-

① $D^n(x^n) = \lfloor n$

② $D^n\{(ax+b)^m\} = \begin{cases} \dfrac{\lfloor m}{\lfloor m-n}(ax+b)^{m-n} \times a^n & \text{when } m>0, m>n \\ = 0 & \text{when } m>0, m<n \\ = \lfloor n . a^n & \text{when } m=n \end{cases}$

③ $D^n\left\{\dfrac{1}{(ax+b)^m}\right\} = (-1)^n \dfrac{\lfloor m+n-1}{\lfloor m-1} \times \dfrac{a^n}{(ax+b)^{m+n}}$

④ $D^n(e^{mx}) = m^n . e^{mx}$ ⑤ $D^n(a^{mx}) = (m\log_e a)^n . a^{mx}$

⑥ $D^n\{\log(ax+b)\} = (-1)^{n-1} \dfrac{\lfloor n-1}{(ax+b)^n} \times a^n$

⑦ $D^n\{\sin(ax+b)\} = a^n \sin\left(\dfrac{n\pi}{2} + ax+b\right)$

⑧ $D^n\{\cos(ax+b)\} = a^n \cos\left(\dfrac{n\pi}{2} + ax+b\right)$

⑨ $D^n\{(e^{ax} . \sin bx)\} = (a^2+b^2)^{n/2} e^{ax} . \sin\left(bx + n\tan^{-1}\dfrac{b}{a}\right)$

⑩ $D^n\{(e^{ax} . \cos bx)\} = (a^2+b^2)^{n/2} e^{ax} . \cos\left(bx + n\tan^{-1}\dfrac{b}{a}\right)$

Problem questions with Answer

Ex. ① If $y = \log(ax+b)(cx+d)$ then find y_n.

Soln Here $y = \log(ax+b)(cx+d)$

$\therefore y = \log(ax+b) + \log(cx+d)$

Diff. n times w.r.t. to x we get

$y_n = D^n[\log(ax+b) + \log(cx+d)]$

$= D^n \log(ax+b) + D^n \log(cx+d)$

$= \dfrac{(-1)^{n-1} \lfloor n-1 \times a^n}{(ax+b)^n} + \dfrac{(-1)^{n-1} \lfloor n-1 \times c^n}{(cx+d)^n}$

$\therefore y_n = (-1)^{n-1}\lfloor n-1 \left[\frac{a^n}{(ax+b)^n} + \frac{c^n}{(cx+d)^n}\right]$ Ans

Ex. (2) If $y = \sin 6x \cdot \cos 4x$ then find y_n

Soln. Here $y = \sin 6x \cdot \cos 4x = \frac{1}{2}(2\sin 6x \cdot \cos 4x)$

$\therefore y = \frac{1}{2}[\sin 10x + \sin 2x)$

Diff. n times w.r.to x we get

$y_n = \frac{1}{2}\left[D^n(\sin 10x) + D^n(\sin 2x)\right]$

$\therefore y_n = \frac{1}{2}\left[10^n \cdot \sin\left(\frac{n\pi}{2} + 10x\right) + 2^n \sin\left(\frac{n\pi}{2} + 2x\right)\right]$ Ans

Ex. (3) If $y = \frac{1}{1-5x+6x^2}$ then find y_n

Soln. Here $y = \frac{1}{1-5x+6x^2} = \frac{1}{6x^2-5x+1}$

or, $y = \frac{1}{6x^2-3x-2x+1} = \frac{1}{3x(2x-1)-1(2x-1)}$

or $y = \frac{1}{(2x-1)(3x-1)}$

or, $y = \left[\frac{2}{2x-1} - \frac{3}{3x-1}\right]$ by partial fraction

Diff. n times w.r.to x we get $\left[\because D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n \lfloor n \cdot a^n}{(ax+b)^{n+1}}\right]$

$y_n = 2 \cdot D^n\left(\frac{1}{2x-1}\right) - 3 \cdot D^n\left(\frac{1}{3x-1}\right)$

$= 2 \cdot \left[\frac{(-1)^n \lfloor n \cdot 2^n}{(2x-1)^{n+1}}\right] - 3\left[\frac{(-1)^n \lfloor n \cdot 3^n}{(3x-1)^{n+1}}\right]$

$\therefore y_n = (-1)^n \lfloor n \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{3^{n+1}}{(3x-1)^{n+1}}\right]$ Ans

Ex. (4) If $y = \frac{1}{x^2+a^2}$ then find y_n

Soln:- Here, $y = \frac{1}{x^2+a^2} = \frac{1}{x^2-i^2a^2} = \frac{1}{(x-ia)(x+ia)}$

or $y = \frac{1}{2ia}\left[\frac{1}{(x-ia)} - \frac{1}{x+ia}\right]$ by partial fraction.

Diff. both sides n times w.r.t. x we get

$$y_n = \frac{1}{2ia}\left[D^n\left(\frac{1}{x-ia}\right) - D^n\left(\frac{1}{x+ia}\right)\right]$$

$$\text{or, } y_n = \frac{1}{2ia}\left[\frac{(-1)^n \lfloor n}{(x-ia)^{n+1}} - \frac{(-1)^n \lfloor n}{(x+ia)^{n+1}}\right] \quad \left[\because D^n\left(\frac{1}{x+a}\right) = \frac{(-1)^n \lfloor n}{(x+a)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right]$$

Now, putting $x = r\cos\theta$, $a = r\sin\theta$ $\therefore x^2+a^2 = r^2$ and $\tan\theta = \frac{a}{x}$ $\therefore \theta = \tan^{-1}\frac{a}{x}$

$$\therefore y_n = \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} - \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n . \lfloor n}{2ia} . \frac{1}{r^{n+1}}\left[(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\right]$$

$$= \frac{(-1)^n . \lfloor n}{2ia.} . \frac{1}{r^{n+1}}\left[\{\cos(n+1)\theta + i\sin(n+1)\theta\} - \{\cos(n+1)\theta - i\sin(n+1)\theta\}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia} . \frac{1}{r^{n+1}}\left[\cancel{\cos(n+1)\theta} + i\sin(n+1)\theta - \cancel{\cos(n+1)\theta} + i\sin(n+1)\theta\right]$$

$$= \frac{(-1)^n \lfloor n}{\cancel{2i}a} . \frac{1}{r^{n+1}} . \cancel{2i}\sin(n+1)\theta = \frac{(-1)^n \lfloor n}{a r^{n+1}} . \sin(n+1)\theta$$

$$\because a = r\sin\theta \therefore r = \frac{a}{\sin\theta}$$

$$\therefore y_n = \frac{(-1)^n . \lfloor n}{a . \left(\frac{a}{\sin\theta}\right)^{n+1}} . \sin(n+1)\theta = \frac{(-1)^n . \lfloor n}{\frac{a^{n+2}}{(\sin\theta)^{n+1}}} . \sin(n+1)\theta$$

$$\therefore y_n = \frac{(-1)^n . \lfloor n}{a^{n+2}} . \sin^{n+1}\theta . \sin(n+1)\theta \text{ where } \theta = \tan^{-1}\frac{x}{a}$$

Ex.(5) If $y = \sin mx + \cos mx$,

Prove that $y_n = m^n\left[1+(-1)^n \sin 2mx\right]^{1/2}$

Soln Here $y = \sin mx + \cos mx$

Diff. n times w.r.t. to x we get

$$y_n = D^n(\sin mx) + D^n(\cos mx)$$

$$= m^n \sin\left(\frac{n\pi}{2}+mx\right) + m^n \cos\left(\frac{n\pi}{2}+mx\right)$$

$$= m^n\left[\sin\left(\frac{n\pi}{2}+mx\right) + \cos\left(\frac{n\pi}{2}+mx\right)\right]$$

$$= m^n\left[\left\{\sin\left(\frac{n\pi}{2}+mx\right)+\cos\left(\frac{n\pi}{2}+mx\right)\right\}^2\right]^{1/2}$$

$$= m^n\left[\sin^2\left(\frac{n\pi}{2}+mx\right)+\cos^2\left(\frac{n\pi}{2}+mx\right) + 2\sin\left(\frac{n\pi}{2}+mx\right)\cdot\cos\left(\frac{n\pi}{2}+mx\right)\right]^{1/2}$$

$$= m^n\left[1 + 2\sin\left(\frac{n\pi}{2}+mx\right)\cdot\cos\left(\frac{n\pi}{2}+mx\right)\right]^{1/2}$$

$$= m^n\left[1+\sin 2\left(\frac{n\pi}{2}+mx\right)\right]^{1/2}$$

$$= m^n\left[1+\sin(n\pi+2mx)\right]^{1/2}$$

$$= m^n\left[1+(-1)^n \sin 2mx\right] \quad \because \sin(n\pi+\theta) = (-1)^n\sin\theta$$

proved