$\vec{b}+\vec{c} = (b_1\vec{i}+b_2\vec{j}+b_3\vec{k}) + (c_1\vec{i}+c_2\vec{j}+c_3\vec{k})$

$= (b_1+c_1)\vec{i} + (b_2+c_2)\vec{j} + (b_3+c_3)\vec{k}$

$$a \times (\vec{b}+\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\therefore a \times (\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$ using (i) and (ii)

Hence the theorem.

Prove that the vectors

$\vec{a} = 4\vec{i} + 5\vec{j} + k$

$\vec{b} = -\vec{j} - \vec{k}$

$\vec{c} = 5\vec{i} + 9\vec{j} + 4\vec{k}$ are Coplaner.

Solution :- We know that the vectors $\vec{a}, \vec{b}, \vec{c}$ will be Coplaner if

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 5 & 1 \\ 0 & -1 & -1 \\ 5 & 9 & 4 \end{vmatrix}$$

$= 4 \times (-4+9) + 5 \times (-5-0) + 1 \times (0+5)$

$= 20 - 25 + 5 = 0$ Hence given vectors are Coplaner.

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Product of three or four vectors.

THEOREM :- A necessary and sufficient Condition that the three non parallel non-zero vector $\vec{a}, \vec{b}, \vec{c}$ be Coplanar is $[\vec{a}\,\vec{b}\,\vec{c}] = 0$.

proof :- The Condition is necessary: To show that if three non parallel, non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ are Coplaner then their scalar triple product $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

Let the three vectors $\vec{a}, \vec{b}, \vec{c}$ be Coplaner. Now the Cross product $\vec{b} \times \vec{c}$ represents a vector perpendicular to the plane Contaning vector $\vec{b} \times \vec{c}$

Also a lies in the plane of $\vec{b}$ & $\vec{c}$ therefore $\vec{b} \times \vec{c}$ is perpendicular to $\vec{a}$ also and so the dot product of $\vec{a}$ and $\vec{b} \times \vec{c}$ or $[\vec{a}\,\vec{b}\,\vec{c}]$ show that if $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ i.e if $\vec{a} \cdot (\vec{b} \times \vec{c})$ is zero then $[\vec{a}\,\vec{b}\,\vec{c}]$ are Coplaner.

Let $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ i.e $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

It implies that $\vec{a}$ and $\vec{b} \times \vec{c}$ are perpendicular to each other and so $\vec{b} \times \vec{c}$ is perpendicular to $\vec{a}$, but $\vec{b} \times \vec{c}$ is perpendicular to the plane Containg the vector $\vec{b}$ and $\vec{c}$

THEOREM :- To prove that

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

proof :- Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$b = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

and $c = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{------- (i)}$$

&

$$\vec{a} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{------- (ii)}$$

Th:- Find the equation of normal to the curve $\frac{l}{r} = 1 + e\cos\theta$ at a given point α. 2

Proof:- Let the eqn of the conic be $\frac{l}{r} = 1 + e\cos\theta$ and α be the vectorial angle of the pt. of contact. Now the eqn of the chord of the conic through the pts whose vectorial angles are $\alpha+\beta$ and $\alpha-\beta$ is

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha)\sec\beta \quad \text{--- (I)}$$

$\therefore$ the equation of tangent at pt α is obtained by putting $\beta = 0$ in eqn (I)

viz. $$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha) \quad \text{--- (II)}$$

Eqn of any line perpr to the tangent (II) is of the form

$$\frac{\lambda}{r} = e\cos\left(\theta+\frac{\pi}{2}\right) + \cos\left(\theta+\frac{\pi}{2}-\alpha\right)$$

$$\Rightarrow \frac{\lambda}{r} = -e\sin\theta - \sin(\theta-\alpha) \quad \text{--- (III)}$$

This will be required equation of the normal provided λ be so chosen that it will pass through the pt. of contact given by

$\theta = \alpha$ and $\frac{l}{r} = 1 + e\cos\theta$

Hence we must have

$$\frac{\lambda(1+e\cos\alpha)}{l} = -e\sin\alpha$$

$$\therefore \lambda = \frac{-l\cdot e\cdot\sin\alpha}{1+e\cdot\cos\alpha}$$

putting the value of λ in (III) we get

$$\frac{1}{r}\frac{(-le\sin\alpha)}{1+e\cos\alpha} = -[e\sin\theta + \sin(\theta-\alpha)]$$

$$\frac{l}{r}\left[\frac{e\sin\alpha}{1+e\cos\alpha}\right] = e\sin\theta + \sin(\theta-\alpha)$$

which is the equation of normal to the curve $\frac{l}{r} = 1+e\cos\theta$ at pt. α

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Ex-1. The normal at the point $(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again in the point $(at_2^2, 2at_2)$ prove that $t_2 = -t_1 - \frac{2}{t_1}$

Ans.

The eqn of the normal at the point $(at_1^2, 2at_1)$ is

$$y = -t_1 x + 2at_1 + at_1^3$$

∵ the normal passes through the point $(at_2^2, 2at_2)$ we have

$$2at_2 = -t_1 \cdot at_2^2 + 2at_1 + at_1^3$$

$$\Rightarrow 2a(t_1 - t_2) = at_1 t_2^2 - at_1^3$$

$$= at_1(t_2^2 - t_1^2)$$

$$= at_1(t_2 + t_1)(t_2 - t_1)$$

$$\Rightarrow 2a\cancel{(t_1 - t_2)} = -at_1(t_1 + t_2)\cancel{(t_1 - t_2)}$$

$$\therefore 2\cancel{a} = -\cancel{a}t_1(t_1 + t_2)$$

$$\text{i.e. } 2 = -t_1(t_1 + t_2)$$

$$\therefore t_1 + t_2 = -\frac{2}{t_1}$$

$$\therefore t_2 = -t_1 - \frac{2}{t_1}$$ proved.

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